AD-A122 257 NUMERICAL SOLUTIONS USING ADJOINT VARIATIONAL FORMULATION TO STRESS WAVE..(U) ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND WATERVLIET NY L.. 1/1

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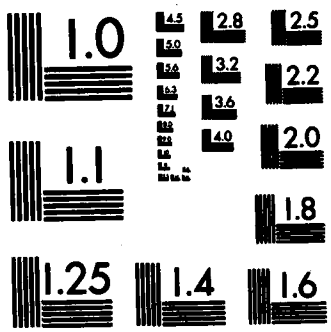

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TECHNICAL REPORT ARLCB-TR-82028

NUMERICAL SOLUTIONS USING ADJOINT VARIATIONAL FORMULATION TO STRESS WAVE PROBLEMS

C. N. Shen
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September 1982

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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AMCMS No. 611102H600011

DA Project No. 1L161102AH60

PRON No. 1A2250041A1A

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER ARLCB-TR-82028	2. GOVT ACCESSION NO. AD-A122 257	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) NUMERICAL SOLUTIONS USING ADJOINT VARIATIONAL FORMULATION TO STRESS WAVE PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) C. N. Shen and J. J. Wu		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Armament Research & Development Command Benet Weapons Laboratory, DRDAR-LCB-TL Watervliet, NY 12189		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 611102H600011 DA Project No. 1L161102AH60 PRON No. 1A2250041A1A
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Research & Development Command Large Caliber Weapon Systems Laboratory Dover, NJ 07801		12. REPORT DATE September 1982
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Presented at 1982 Army Numerical Analysis and Computers Conference, US Army Engineers Waterways Experiment Station, Vicksburg, MS, 3-4 February 1982. Published in proceedings of the conference.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Adjoint Variational Principle
Initial and Boundary Value Problems
Wave Equation
Spline Function
Finite Elements
Assembly of Matrices
Matrix Row Elimination

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report deals with the numerical implementation of an involved analysis in conjunction with cubic Hermite polynomials as the approximate functions. The specific example used for numerical results is the longitudinal stress wave of a uniform bar.

(CONT'D ON REVERSE)

20. ABSTRACT (CONT'D)

First, the adjoint principle associated with this problem is stated. It is followed by the discretized counterparts in spatial and temporal dimensions. The procedures involving the assemblage of the "mass" and "stiffness" matrices in the two dimensions are described. Due to the null variations of some adjoint variables, certain rows of the matrices are eliminated. Because certain variables are known at the boundaries, the unknown variables for the next interval of time can be computed by inversion of a band matrix in terms of their present values.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

INTRODUCTION

A well known advantage of variational solution formulation to boundary value problems is that the differentiability requirements of the approximate solutions can be relaxed. For initial value problems, however, this advantage is somewhat diminished by the complication due to the appearance of the farther end condition. This complication can be eliminated by the use of an adjoint variational principle as we have demonstrated for a simple initial value problem in a previous paper.[1] The more involved analysis for the mixed initial-boundary value problem has also been worked out.

The purpose of this report is to employ the adjoint variational principle in the form of finite element formulation for solving the stress wave problems. The hyperbolic partial differential equation governing the motion is second order both in spatial and time domains.

$$Ly(x,t) + Q(x,t) = 0 \tag{1}$$

where

$$Ly = (\alpha y_t)_t + (\ell y_x)_x \tag{2}$$

We seek explicitly the numerical transient solutions of y, y_t, y_x and y_{xt} for assigned boundary and initial conditions. The term y_x will give the stress wave in a longitudinal bar. The study is the extension of previous work on initial and boundary value problems.[1]

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[1] Shen, C. N., "Method of Solution For Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

INTEGRAL OF BILINEAR EXPRESSION

The integral of a bilinear expression for a two-dimensional problem having second order partial derivatives in both space and time can be written as

$$I = \int_{x_o}^{x_b} \int_{t_o}^{t_b} \Omega[y(x,t),\bar{y}(x,t)]dtdx \tag{3}$$

where $\Omega[y,\bar{y}]$ is a given bilinear expression in the form

$$\Omega[y,\bar{y}] = \alpha y_t \bar{y}_t + \ell y_x \bar{y}_x \tag{4}$$

The quantity $\bar{y}$ is the adjoint of y and the subscripts t and x indicate the partial derivatives of the functions y and $\bar{y}$.

Two different forms of integrals and end conditions can be obtained from Eq. (4). The first form is obtained by integrating by parts on the adjoint variable.

$$I = -\int_{x_o}^{x_b} \int_{t_o}^{t_b} \bar{y}Lydtdx + \int_{x_o}^{x_b} \alpha y_t \bar{y}\Big|_{t_o}^{t_b} dx + \int_{t_o}^{t_b} \ell y_x y\Big|_{x_o}^{x_b} dt \tag{5}$$

where Ly is given in Eq. (2).

In addition, we can perform integration on the original variable to give

$$I = -\int_{x_o}^{x_b} \int_{t_o}^{t_b} y\overline{Ly}dtdx + \int_{x_o}^{x_b} \alpha \bar{y}_t y\Big|_{t_o}^{t_b} dx + \int_{t_o}^{t_b} \ell y_x y\Big|_{x_o}^{x_b} dt \tag{6}$$

where

$$\overline{Ly} = (\alpha \bar{y}_t)_t + (\ell \bar{y}_x)_x \tag{7}$$

In a previous paper[1] we show that the bilinear concomitant D has to be identically zero, i.e.,

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[1]Shen, C. N., "Method of Solution For Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

$$D = \int_{x_o}^{x_b} \int_{t_o}^{t_b} y L \bar{y} dt dx - \int_{x_o}^{x_b} \int_{t_o}^{t_b} y \overline{L y} dt dx \tag{8}$$

By equating Eqs. (5) and (6) and solving for D in Eq. (8), we are converting the double integral into two single integrals in terms of the initial and boundary conditions.

We can express the quantity D as the sum of two parts for end conditions as D_1 and D_2. Thus one defines

$$D = D_1 + D_2 \tag{9}$$

The terms in D_1 involve the initial conditions of y and $\bar{y}$ as

$$\begin{aligned} D_1 &= \int_{x_o}^{x_b} \{\alpha y_t \bar{y}\Big|_{t_o}^{t_b} - \alpha \bar{y}_t y\Big|_{t_o}^{t_b}\} dx \\ &= \int_{x_o}^{x_b} \{\alpha_b (y_{tb}\bar{y}_b - \bar{y}_{tb} y_b) - \alpha_o (y_{to}\bar{y}_o - \bar{y}_{to} y_o)\} dx \end{aligned} \tag{10}$$

The terms in D_2 involve the boundary conditions of y and $\bar{y}$ as

$$\begin{aligned} D_2 &= \int_{t_o}^{t_b} \{\ell y_x \bar{y}\Big|_{x_o}^{x_b} - \ell \bar{y}_x y\Big|_{x_o}^{x_b}\} dt \\ &= \int_{t_o}^{t_b} \{\ell_b (y_{xb}\bar{y}_b - \bar{y}_{xb} y_b) - \ell_o (y_{xo}\bar{y}_o - \bar{y}_{xo} y_o)\} dt \end{aligned} \tag{11}$$

In order that $D \equiv 0$ in Eq. (9) it is sufficient that

$$D_1 \equiv 0 \tag{12a}$$

and

$$D_2 \equiv 0 \tag{12b}$$

END CONDITIONS FOR THE ADJOINT SYSTEMS

In order to satisfy the two requirements in Eq. (12), we separate them in two parts. Let us consider first the time domain and assume that the adjoint variables are assigned as

$$\bar{y}_b = y_o \quad , \quad \bar{y}_o = y_b$$

$$\bar{y}_{tb} = -\alpha_b^{-1}\alpha_o\, y_{to} \quad , \quad \bar{y}_{to} = -\alpha_o^{-1}\alpha_b\, y_{tb}$$

$$\alpha_b \neq 0 \qquad \alpha_o \neq 0$$

The above adjoint initial conditions satisfy the requirement that D...
Eq. (10). Now we turn to the spatial domain and assume that the ad...
variables are

$$\bar{y}_b = \beta y_b \qquad \bar{y}_o = \beta y_o$$

$$\bar{y}_{xb} = \beta y_{xb} \qquad \bar{y}_{xo} = \beta y_{xo}$$

The above adjoint boundary conditions satisfy the requirement that ...
Eq. (11), with β as a constant.

By giving the appropriate values of these adjoint variables in ter...
the original variables, one may find that the requirement $D \equiv 0$ can be satisfied. This leads to the result[1] previously found as

$$J[\bar{y},y] = \int_{t_o}^{t_b}\int_{x_o}^{x_b} \bar{Q}y\,dt\,dx + \int_{t_o}^{t_b}\int_{x_o}^{x_b} \bar{y}(Q+1\_\ dt\,dx = 0 \qquad (1\ldots$$

FIRST VARIATION

By taking variation on Eq. (18) we have

$$\delta J = \delta J[\delta\bar{y}] + \delta J[\delta y]$$

$$= \int_{t_o}^{t_b}\int_{x_o}^{x_b} \delta y(\overline{\bar{L}y})\,dt\,dx + \int_{t_o}^{t_b}\int_{x_o}^{x_b} \bar{y}(L\delta y)\,dt\,dx = 0 \qquad (\ldots$$

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[1]Shen, C. N., "Method of Solution For Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

where

$$\delta J[\delta\bar{y}] = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \delta y(Ly+\bar{Q})dtdx \tag{20}$$

and

$$\delta J[\delta y] = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \delta y(\bar{L}\bar{y}+\bar{Q})dtdx \tag{21}$$

Since $D \equiv 0$ in Eq. (8) the variation δD should be zero

$$\delta D = \delta D[\delta\bar{y}] + \delta D[\delta y] = 0 \tag{22}$$

Since the variations $\delta\bar{y}$ and δy are independent, then

$$\delta D[\delta y] = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \bar{y}(L\delta y)dtdx - \int_{t_o}^{t_b} \int_{x_o}^{x_b} \delta y(\bar{L}\bar{y})dtdx = 0 \tag{23}$$

which is the same as the last two terms in Eq. (19) which vanish. Thus

$$\delta J = \delta J[\delta\bar{y}] + \delta J[\delta y] = 0 \tag{24}$$

Since the variations $\delta\bar{y}$ and δy are independent

$$\delta J[\delta\bar{y}] = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \delta\bar{y}(Ly+Q)dtdx = 0 \tag{25}$$

where Ly is given in Eq. (2) and contains higher derivatives than the first partials in y. It is intended to include only lower order partial differentiation in y. This can be achieved by considering the variations of the bilinear expression I given by Eqs. (3) and (4) as

$$\delta I[\delta\bar{y}] = \int_{t_o}^{t_b} \int_{x_o}^{x_b} [\alpha y_t \delta\bar{y}_t + \ell y_x \delta\bar{y}_x]dtdx \tag{26}$$

A different form of the above variation can be obtained from Eq. (5) as

$$\delta I[\delta\bar{y}] = -\int_{x_o}^{x_b} \int_{t_o}^{t_b} \delta\bar{y}Lydtdx + \int_{x_o}^{x_b} \delta\bar{y}\alpha y_t\Big|_{t_o}^{t_b} dx + \int_{t_o}^{t_b} \delta\bar{y}\ell y_x\Big|_{x_o}^{x_b} dt \tag{27}$$

Equating Eqs. (26) and (27), solving for the term containing integral for $\delta\bar{y}Ly$ and substituting into Eq. (25) we have

$$\delta J[\delta\bar{y}] = \int_{x_o}^{x_b} \alpha y_t \delta\bar{y}\Big|_{t_o}^{t_b} dx + \int_{t_o}^{t_b} \ell y_x \delta\bar{y}\Big|_{x_o}^{x_b} dt$$

$$+ \int_{x_o}^{x_b}\int_{t_o}^{t_b} \delta\bar{y}Q\,dt\,dx - \int_{x_o}^{x_b}\int_{t_o}^{t_b} [\alpha y_t \delta\bar{y}_t + \ell y_x \delta\bar{y}_x]\,dt\,dx = 0 \qquad (28)$$

This is the key equation which uses variational principle in solving a mixed initial and boundary value problem for a wave equation.

DISCUSSION OF THE VARIATIONAL EQUATION

Let us discuss the various terms in Eq. (28), the variational formulation of the wave equation, into three parts as follows.

(1) The initial conditions of the original variables are given and variations of the adjoints at the far end are zero. The first term in Eq. (28) contains the product of $y_t\delta\bar{y}$ evaluated at the initial condition $y_{to}\delta y_o$ and at the final condition $y_{tb}\delta\bar{y}_b$. Since the value of y_b is known by Eqs. (13) and (16), $\delta\bar{y}_b = 0$. That is, the variations of the adjoint variable at the far end are zero.

(2) The boundary conditions of the original variables and variation of the adjoints can be determined. The second term in Eq. (28) is the boundary term involving the variation $\delta\bar{y}$ and the variable y_x. For a longitudinal or a torsional bar the end conditions are

from Eq. (16) for the fixed end

$$y = 0 \qquad \bar{y} = 0 \qquad \delta\bar{y} = 0 \qquad (29)$$

from Eq. (17) for the free end

$$y_x = 0 \qquad \bar{y}_x = 0 \qquad \delta\bar{y}_x = 0 \tag{30}$$

The variations in the adjoint variables shown in the last column coincide with the same end conditions in the original variables given in the first column, whether they are on the left or the right boundary.

(3) Interior Region - The last two terms in Eq. (28) give the interior region where the forcing function Q, the adjoint variations $\delta\bar{y}$, $\delta\bar{y}_t$, and $\delta\bar{y}_x$ and the variables y_t, and y_x are shown. No second order partial of y with respect to x is present. Thus the variables that are needed for the computation are y, y_t, and y_x. This requires a c' continuity in both spatial and time domain.

TRANSFORMATION OF COORDINATES

The integral signs in Eq. (28) can be converted into summation signs if discrete intervals for integration are used. We may take some scale factors to nondimensionalize the problem by giving

$$t_o = 0 \quad , \quad t_b = 1 \qquad 0 < t < 1 \tag{31}$$

$$x_o = 0 \quad , \quad x_b = 1 \qquad 0 < x < 1 \tag{32}$$

Moreover, Eq. (28) can be discretized by letting

$$\xi = Ht-i+1 \qquad 0 < \xi < 1 \qquad i = 1,2,\ldots,H \tag{33}$$

$$\eta = Kx-j+1 \qquad 0 < \eta < 1 \qquad i = 1,2,\ldots,K \tag{34}$$

where H and K are number of intervals for t and x respectively. Thus the partial derivatives are:

$$y_t = \frac{\partial y}{\partial t} = H \frac{\partial y}{\partial \xi} = Hy_\xi \tag{35}$$

$$y_x = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial \eta} = Ky_\eta \tag{36}$$

Use of Eqs. (28), (31) through (36) then leads to

$$\begin{aligned} 0 &= \delta J[\delta \bar{y}] \\ &= \sum_{j=1}^{K} \frac{H}{K} \int_0^1 \alpha y_\xi^{(i,j)} \delta y^{(i,j)} d\eta \Big|_{t_o}^{t_b} \\ &+ \sum_{i=1}^{H} \frac{K}{H} \int_0^1 \ell y_\eta^{(i,j)} \delta y^{(i,j)} d\xi \Big|_{x_o}^{x_b} \\ &+ \sum_{j=1}^{K} \sum_{i=1}^{H} \frac{1}{HK} \int_0^1 \int_0^1 Q \delta \bar{y}^{(i,j)} d\xi d\eta \end{aligned}$$

$$- \sum_{j=1}^{K} \sum_{i=1}^{H} \{\frac{H}{K} \int_0^1 \int_0^1 \alpha y_\xi^{(i,j)} \delta \bar{y}_\xi^{(i,j)} d\xi d\eta + \frac{K}{H} \int_0^1 \int_0^1 \ell y_\eta^{(i,j)} \delta \bar{y}_\eta^{(i,j)} d\xi d\eta\} \tag{37}$$

SPLINE FUNCTION

We may express the variables $y^{(i,j)}$ and $\delta\bar{y}^{(i,j)}$ in Eq. (37) in terms of the (1x16) spline function $a^T(\xi,\eta)$ and the (16x1) node point function $Y^{(i,j)}$ as follows.

$$y^{(i,j)}(\xi,\eta) = a^T(\xi,\eta) Y^{(i,j)} \tag{38}$$

where

$$a^T(\xi,\eta) = \{[a^1(\xi,\eta)]^T \; [a^2(\xi,\eta)]^T \; [a^3(\xi,\eta)]^T \; [a^4(\xi,\eta)]^T\} \tag{39}$$

and

$$\delta\bar{y}^{(i,j)}(\xi,\eta) = a^T(\xi,\eta) \delta \bar{Y}^{(i,j)} \tag{40}$$

A typical term for a product can be written as

$$\delta\bar{y}(i,j)y(i,j) = [\delta\bar{Y}^{(i,j)}]^T a(\xi,\eta) a^T(\xi,\eta) Y^{(i,j)} \tag{41}$$

Thus Eq. (37) becomes

$$\begin{aligned}
\delta J(\delta\bar{y}) = {} & \sum_{j=1}^{K} \delta\bar{Y}^{(t_b,j)}]^T P_{0\xi}(t_b) Y^{(t_b,j)} \\
& - \sum_{j=1}^{K} \delta\bar{Y}^{(t_o,j)}]^T P_{0\xi}(t_o) Y^{(t_o,j)} \\
& + \sum_{i=1}^{H} [\delta\bar{Y}^{(i,x_b)}]^T P_{0\eta}(x_b) Y^{(i,x_b)} \\
& - \sum_{i=1}^{H} \delta\bar{Y}^{(i,x_o)}]^T P_{0\eta}(x_o) Y^{(i,x_o)} \\
& + \sum_{j=1}^{K} \sum_{i=1}^{H} [\delta\bar{Y}^{(i,j)}]^T q^{(i,j)} \\
& - \sum_{j=1}^{K} \sum_{i=1}^{H} [\delta\bar{Y}^{(i,j)}]^T P^{(i,j)} Y^{(i,j)} = 0
\end{aligned} \tag{42}$$

where the coefficient P contains integrals involving the spline functions $a(\xi,\eta)$ and its partial derivatives as given in a previous paper.[1]

GRID SYSTEMS FOR FINITE ELEMENT

We take a finite element represented by the (16x1) vector $Y^{(i,j)}$ which has a grid of four (4x1) vectors $Y_1^{(i,j)}$ through $Y_4^{(i,j)}$, thus

$$Y^{(i,j)} = \{[Y_1^{(i,j)}]^T\ [Y_2^{(i,j)}]^T\ [Y_3^{(i,j)}]^T\ [Y_4^{(i,j)}]^T\} \tag{43}$$

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[1] Shen, C. N., "Method of Solution For Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

Each of the (4x1) vectors has four components, consisting of the function, its first partials in both directions, and its mixed partial, as shown in Figure 1.

These vectors are,

$$Y_1(i,j) = \begin{bmatrix} y(\xi_i,\eta_j) \\ y_\xi(\xi_i,\eta_j) \\ y_\eta(\xi_i,\eta_j) \\ y_{\xi\eta}(\xi_i,\eta_j) \end{bmatrix} \qquad Y_3(i,j) = \begin{bmatrix} y(\xi_i,\eta_{j+1}) \\ y_\xi(\xi_i,\eta_{j+1}) \\ y_\eta(\xi_i,\eta_{j+1}) \\ y_{\xi\eta}(\xi_i,\eta_{j+1}) \end{bmatrix}$$

$$Y_2(i,j) = \begin{bmatrix} y(\xi_{i+1},\eta_j) \\ y_\xi(\xi_{i+1},\eta_j) \\ y_\eta(\xi_{i+1},\eta_j) \\ y_{\xi\eta}(\xi_{i+1},\eta_j) \end{bmatrix} \qquad Y_4(i,j) = \begin{bmatrix} y(\xi_{i+1},\eta_{j+1}) \\ y_\xi(\xi_{i+1},\eta_{j+1}) \\ y_\eta(\xi_{i+1},\eta_{j+1}) \\ y_{\xi\eta}(\xi_{i+1},\eta_{j+1}) \end{bmatrix} \tag{44}$$

We use the vertical direction for the temporal domain. If we increase the row index from i to i+1, then the grid point shifts down by one step and the following holds

$$Y_1(i+1,j) = Y_2(i,j) \qquad Y_3(i+1,j) = Y_4(i,j) \tag{45}$$

If we increase the column index from j to j+1, then the grid point shifts to the right by one step and one obtains

$$Y_1(i,j+1) = Y_3(i,j) \qquad Y_2(i,j+1) = Y_4(i,j) \tag{46}$$

Figure 2 shows the relationship of the grid system by assembly of finite elements in the horizontal direction, which is in the spatial domain.

ASSEMBLY OF MATRICES

In order to solve Eq. (42) by finite element method, assembly of matrices from local form into global form is necessary. For instance, the last term of Eq. (42) is taken as $-\delta J_p(\delta y)$. Then

$$\delta J_p(\delta \bar{y}) = \sum_{j=1}^{K} \sum_{i=1}^{H} [\delta \bar{Y}(i,j)]^T P(i,j) Y(i,j) \tag{47}$$

Since we know that the interval in time can be made as small as possible, with $H = 1$, we have

$$\begin{aligned} \delta J_p(\delta \bar{y}) &= \sum_{j=1}^{K} [\delta \bar{Y}(1,j)]^T P(1,j) Y(1,j) \\ &= \sum_{j=1}^{K} \{[\delta \bar{Y}_1(1,j)]^T [\delta \bar{Y}_2(1,j)]^T [\delta \bar{Y}_3(1,j)]^T [\delta \bar{Y}_4(1,j)]^T\} \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} \begin{bmatrix} Y_1(1,j) \\ Y_2(1,j) \\ Y_3(1,j) \\ Y_4(1,j) \end{bmatrix} \end{aligned} \tag{48}$$

It is noted from Figure 2 that the variables can be indexed as

$$Y_3(1,j) = Y_1(1,j+1) = Y_{2j+1} \tag{49}$$

$$Y_4(1,j) = Y_2(1,j+1) = Y_{2j+2} \tag{50}$$

$$j=0,1,\ldots k$$

For $j = 0$,

$$Y_1(1,1) = Y_1 \quad , \quad Y_2(1,1) = Y_2 \tag{51}$$

For $j = k = 5$

$$Y_3(1,5) = Y_{11} \quad , \quad Y_4(1,5) = Y_{12} \tag{52}$$

Also from Figure 2, the adjoint variations are

$$\delta\bar{Y}_3(1,j) = \delta\bar{Y}_1(1,j+1) = \delta\bar{Y}_{2j+1} \quad (53)$$

$$\delta\bar{Y}_4(1,j) = \delta\bar{Y}_2(1,j+1) = \delta\bar{Y}_2j+1 \quad (54)$$

For j = 0,

$$\delta\bar{Y}_1(1,j) = \delta\bar{Y}_1 \quad , \quad \delta\bar{Y}_2(1,1) = \delta\bar{Y}_2 \quad (55)$$

For j = k = 5,

$$\delta\bar{Y}_3(1,5) = \delta\bar{Y}_{11} \quad , \quad \delta\bar{Y}_4(1,5) = \delta\bar{Y}_{12} \quad (56)$$

Now the local matrices in Eq. (48) can be assembled into a global band matrix shown in Figure 3. Those elements not explicitly written are zeroes in Figure 3.

Since the adjoint variable $\bar{y}_b$ at the far end is assigned in terms of the known initial value y_o, the variation is

$$\delta\bar{y}_b = 0 \quad (57)$$

From Figure 2 we have

$$\delta\bar{Y}_2 = \delta\bar{Y}_4 = \delta\bar{Y}_6 = \delta\bar{Y}_8 = \delta\bar{Y}_{10} = \delta\bar{Y}_{12} = \delta\bar{Y}_{EVEN} = 0 \quad (58)$$

This is equivalent to deleting the even rows of the matrix in Figure 3. The deletion is marked in Figure 4. The number of relationships is reduced to half of the original dimension.

The variables Y_{ODD} in Figure 2 are the initial values of the problem which are supposed to be given. Thus, Y_1, Y_3, Y_5, Y_7, Y_9, Y_{11} are all knowns. The coefficients related to these knowns should eventually be shifted to the right side of the equation.

FURTHER DELETIONS AND KNOWNS

Suppose we have a bar with the fixed end at the left. Then from Eq. (29) one obtains

$$y_o = 0 \tag{59}$$

and

$$\delta\bar{y}_o = 0 \tag{60}$$

The above equations translate to be

$$y^{(2,1)} = 0 \quad \text{(known)} \tag{61}$$

and

$$\delta\bar{y}^{(1,1)} = 0 \quad \text{(deletion)} \tag{62}$$

On the other hand we have a free end at the right. Then Eq. (30) gives

$$y_{xb} = 0 \tag{63}$$

and

$$\delta\bar{y}_{xb} = 0 \tag{64}$$

The above equations yield

$$y_\eta^{(2,6)} = 0 \quad \text{(known)} \tag{65}$$

and

$$\delta y_\eta^{(1,6)} = 0 \quad \text{(deletion)} \tag{66}$$

Figure 5 gives the variation of adjoint variables. It shows two extra zero variations at the first row, $\delta\bar{y}^{(1,1)}$ at the left and $\delta\bar{y}_\eta^{(1,6)}$ at the right. We have also all zero variations on the second row. Figure 6 shows the known and unknown variables. There are two extra known variables in the second row due to boundary conditions, $y^{(2,1)}$ at the left and $y_\eta^{(2,6)}$ at the right. The first row gives all known initial conditions.

CONCLUSIONS

Direct computation of stress, i.e., numerical solution for first spatial derivatives of the displacement can be obtained directly. This is important if the problem has noisy components in the solution of the displacement.

Computation can be made successively, i.e., the final values of the solution at the first stage in time can be used as the initial values of the second period in time. The variations of the adjoint variables at the far end in time for an initial-boundary value problem are zeroes. Deletion of many rows in the assembled matrix is possible. The assembled matrix for computing is reduced to less than half size in linear dimension, from (2nx2n) to (nxn). Hence, a bigger number of intervals in the spatial dimension can be handled. The reduced matrix is a band matrix which makes the storage requirement for computation much easier.

REFERENCES

1. Shen, C. N., "Method of Solution For Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

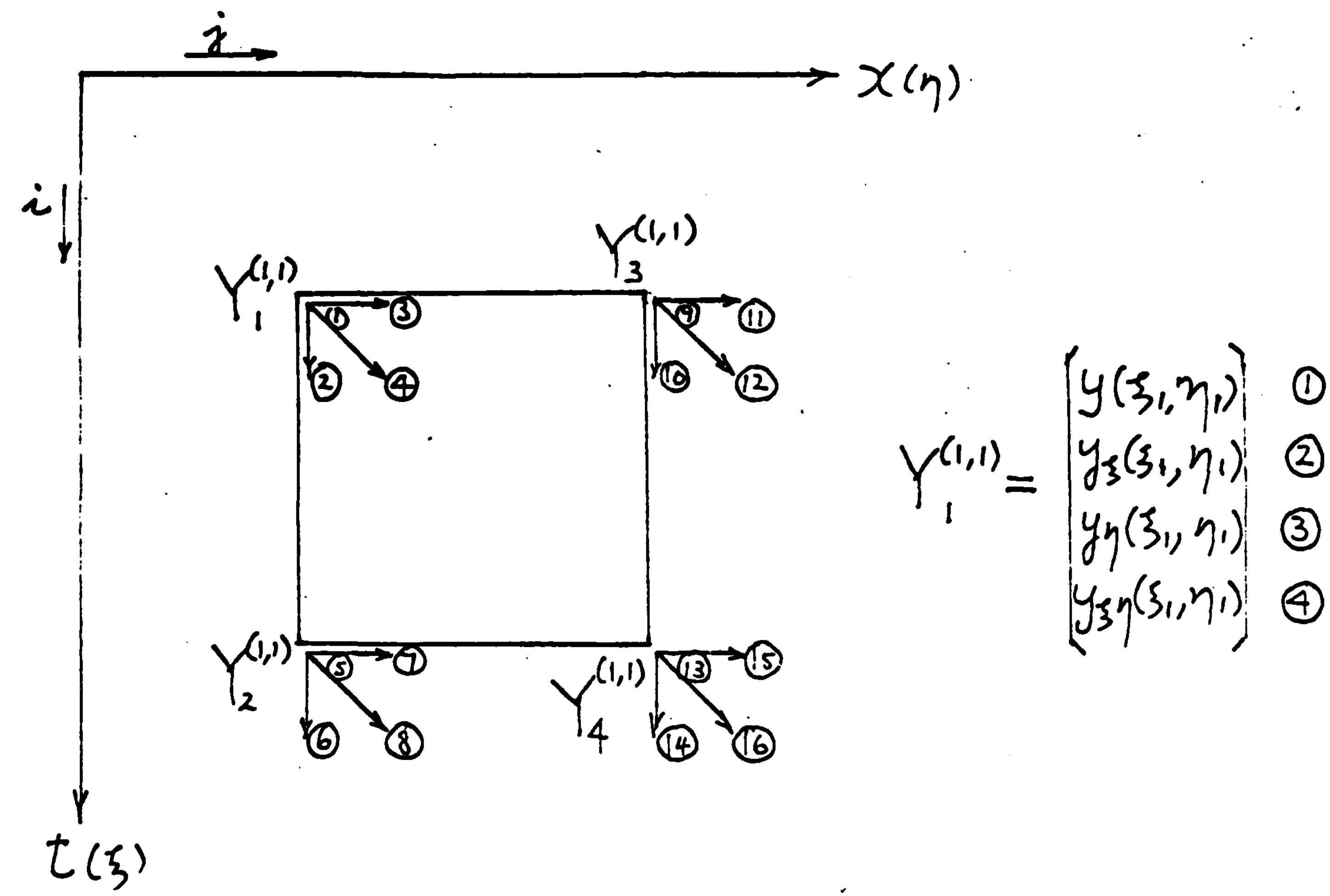

Figure 1. Vectors in a Finite Element.

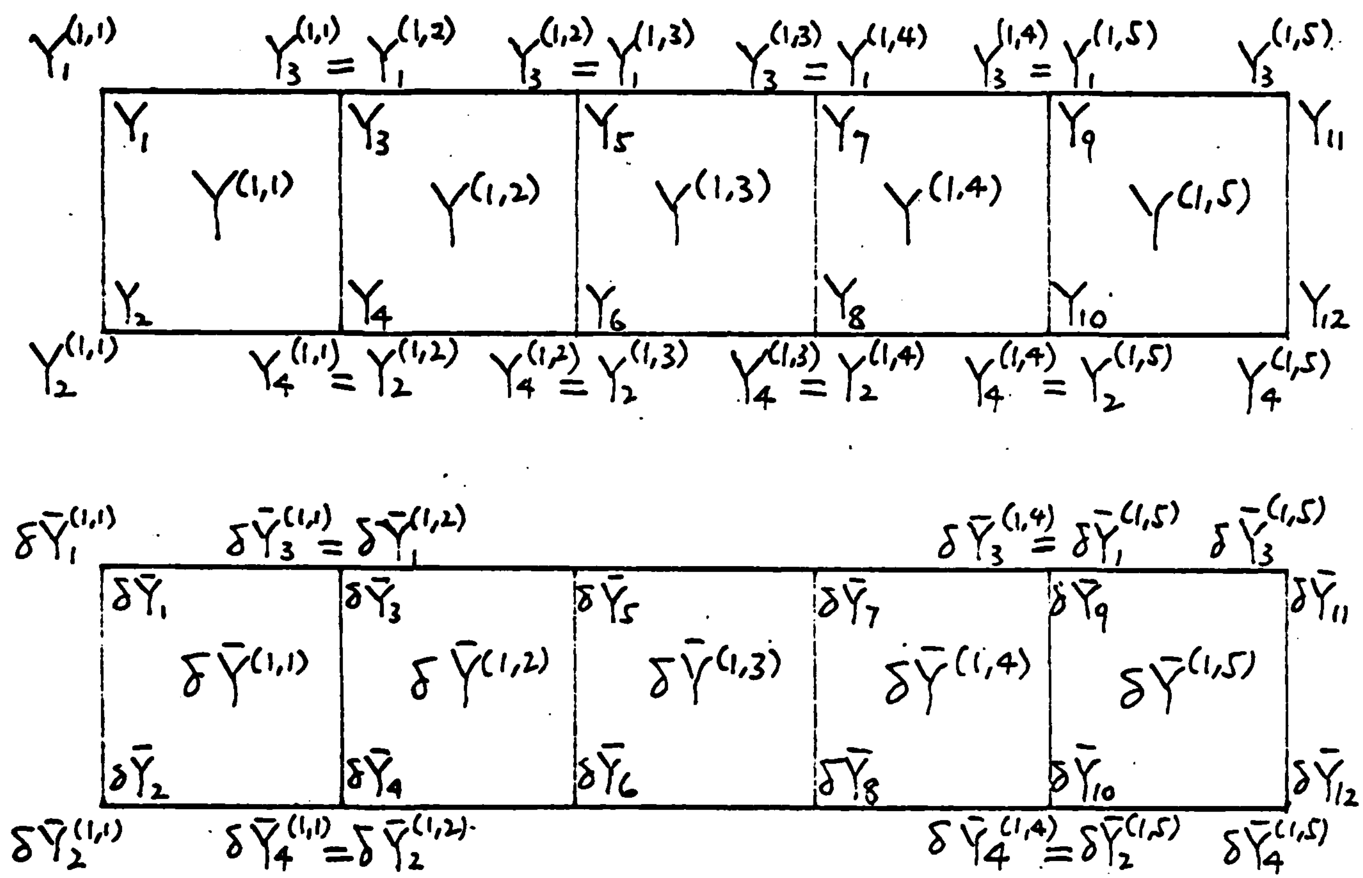

Figure 2. Grid System by Assembly of Finite Elements.

$$
\begin{Bmatrix} \delta\bar{Y}_1 \\ \delta\bar{Y}_2 \\ \delta\bar{Y}_3 \\ \delta\bar{Y}_4 \\ \delta\bar{Y}_5 \\ \delta\bar{Y}_6 \\ \delta\bar{Y}_7 \\ \delta\bar{Y}_8 \\ \delta\bar{Y}_9 \\ \delta\bar{Y}_{10} \\ \delta\bar{Y}_{11} \\ \delta\bar{Y}_{12} \end{Bmatrix}^T
\begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & & & & & & & & \\
P_{21} & P_{22} & P_{23} & P_{24} & & & & & & & & \\
P_{31} & P_{32} & P_{aa} & P_{ab} & P_{13} & P_{14} & & & & & & \\
P_{41} & P_{42} & P_{ba} & P_{bb} & P_{23} & P_{24} & & & & & & \\
 & & P_{31} & P_{32} & P_{aa} & P_{ab} & P_{13} & P_{14} & & & & \\
 & & P_{41} & P_{42} & P_{ba} & P_{bb} & P_{23} & P_{24} & & & & \\
 & & & & P_{31} & P_{32} & P_{aa} & P_{ab} & P_{13} & P_{14} & & \\
 & & & & P_{41} & P_{42} & P_{ba} & P_{bb} & P_{23} & P_{24} & & \\
 & & & & & & P_{31} & P_{32} & P_{aa} & P_{ab} & P_{13} & P_{14} \\
 & & & & & & P_{41} & P_{42} & P_{ba} & P_{bb} & P_{23} & P_{24} \\
 & & & & & & & & P_{31} & P_{32} & P_{33} & P_{34} \\
 & & & & & & & & P_{41} & P_{42} & P_{43} & P_{44}
\end{bmatrix}
\begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \\ Y_{10} \\ Y_{11} \\ Y_{12} \end{Bmatrix}
$$

$\delta\bar{Y}_{EVEN} = 0$ $\quad$ $P_{aa} = P_{11} + P_{33}$ $\quad$ $P_{ab} = P_{12} + P_{34}$ $\quad$ $Y_{ODD} \rightarrow K_{NOV}$

$P_{ba} = P_{21} + P_{43}$ $\quad$ $P_{bb} = P_{22} + P_{44}$

Figure 3. Matrix Assembly Global Form.

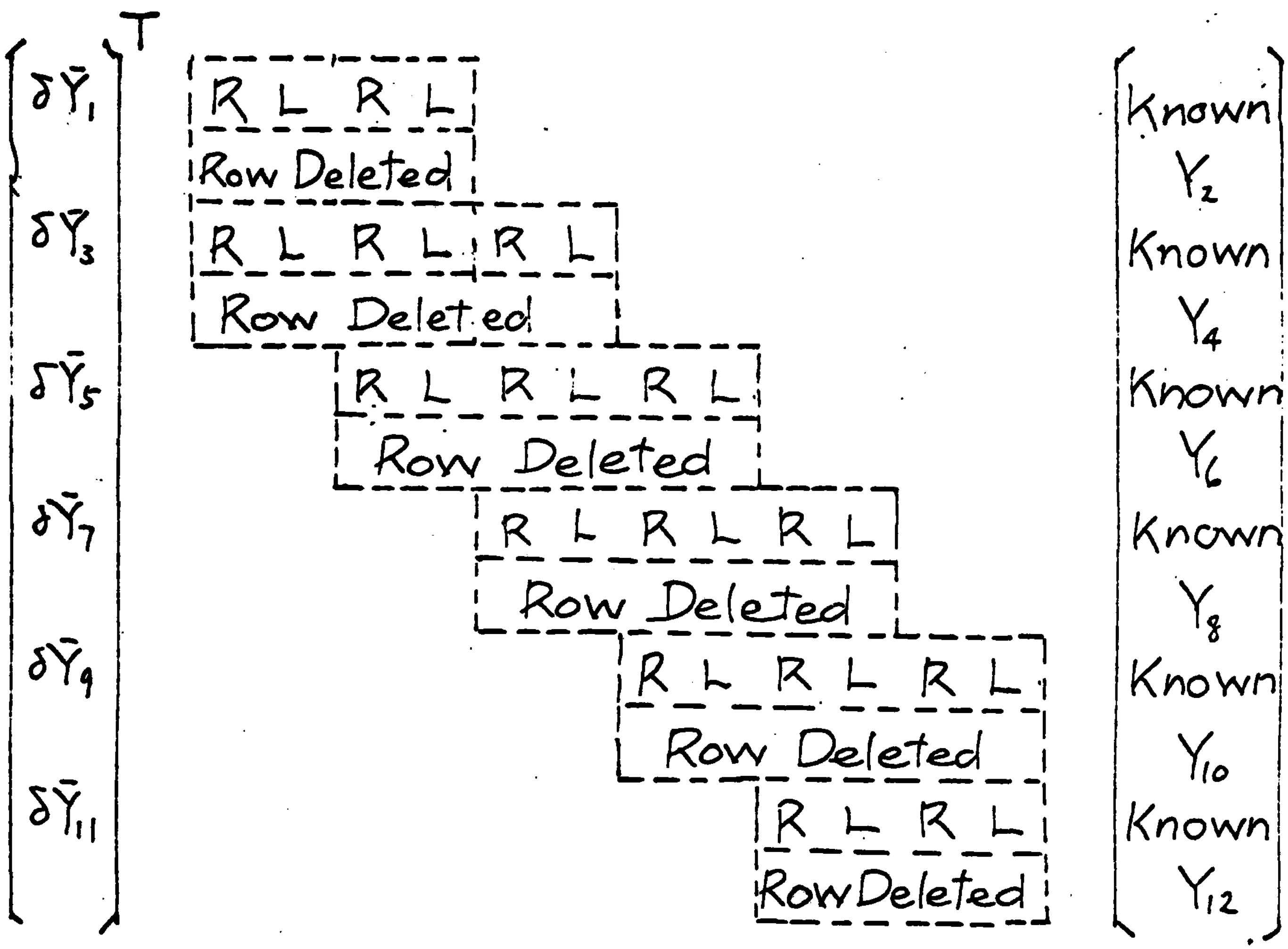

Figure 4. Row Deletion and Knowns.

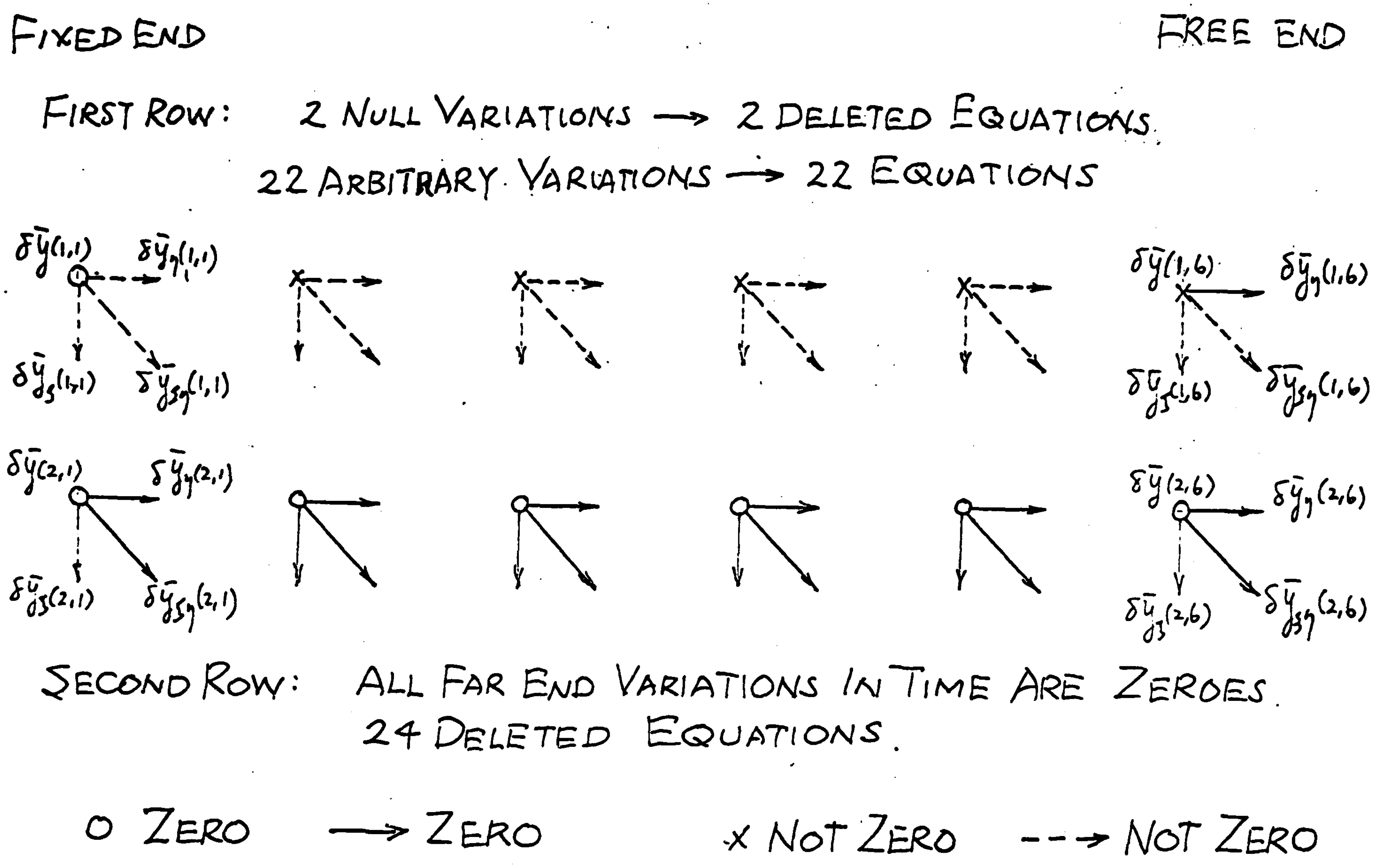

Figure 5. Zero Variation of Adjoint Variables.

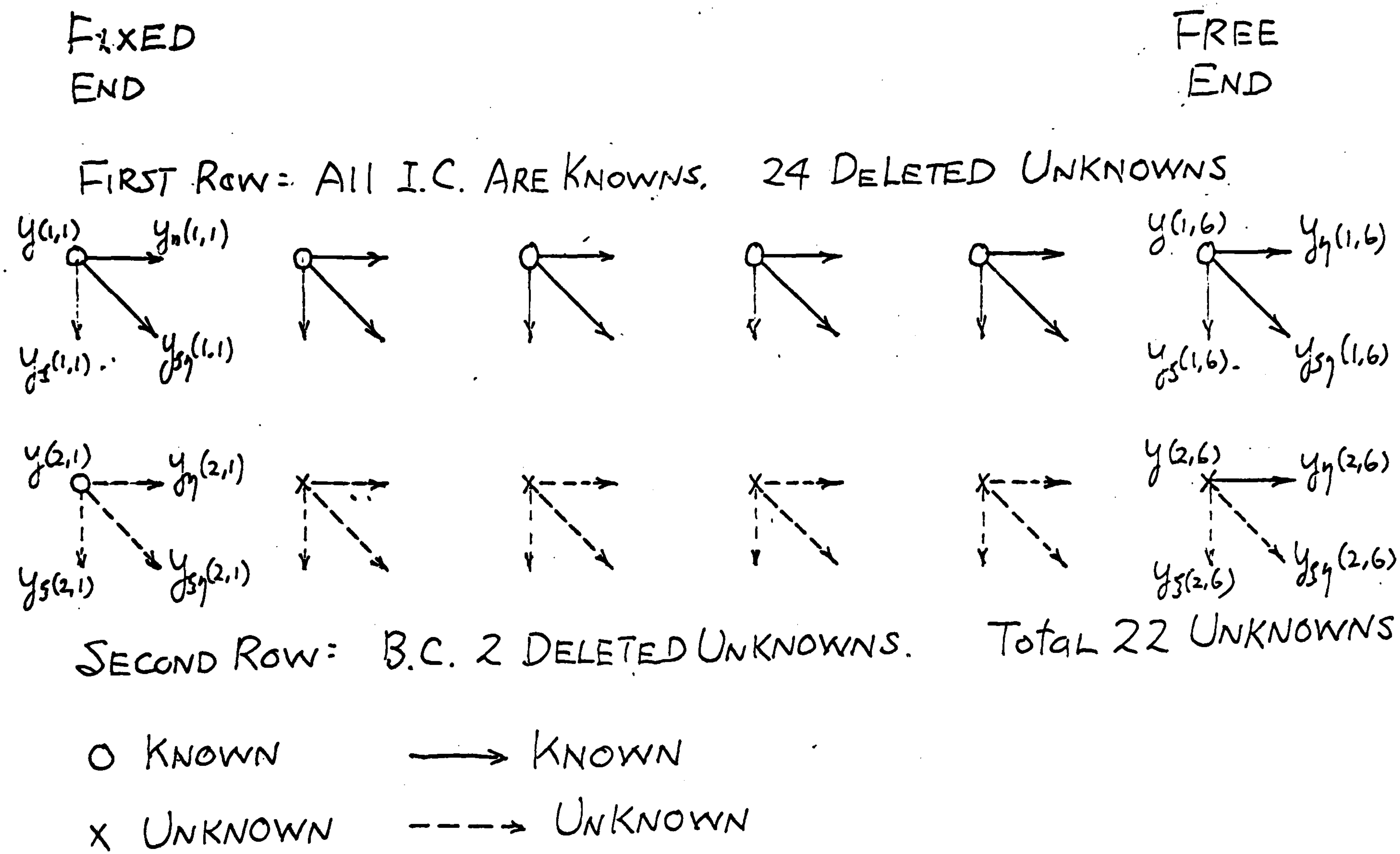

Figure 6. Variables, Known or Unknown.

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